

Recognizing String Graphs in NP

Marcus Schaefer
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Identification des graphes de corde dans NP

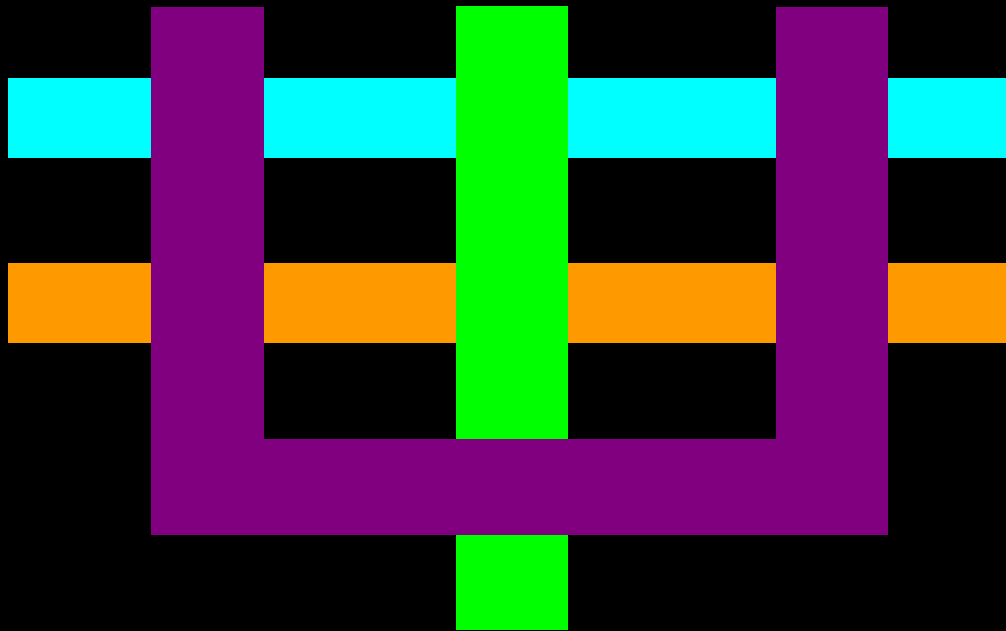
(Recognizing String Graphs in NP)

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The origin of the problem

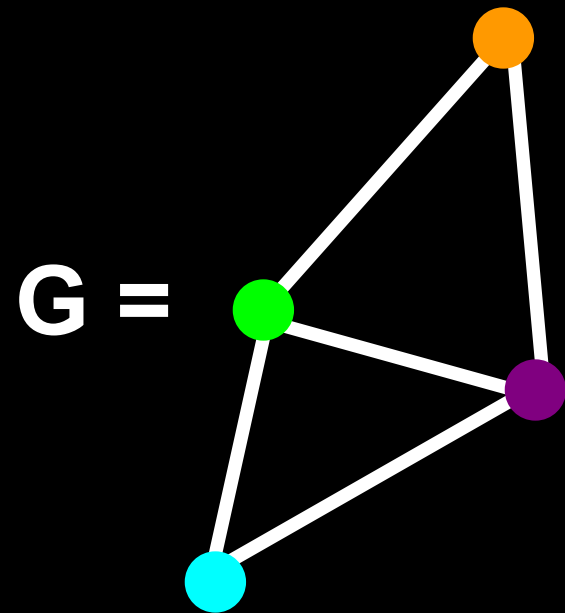
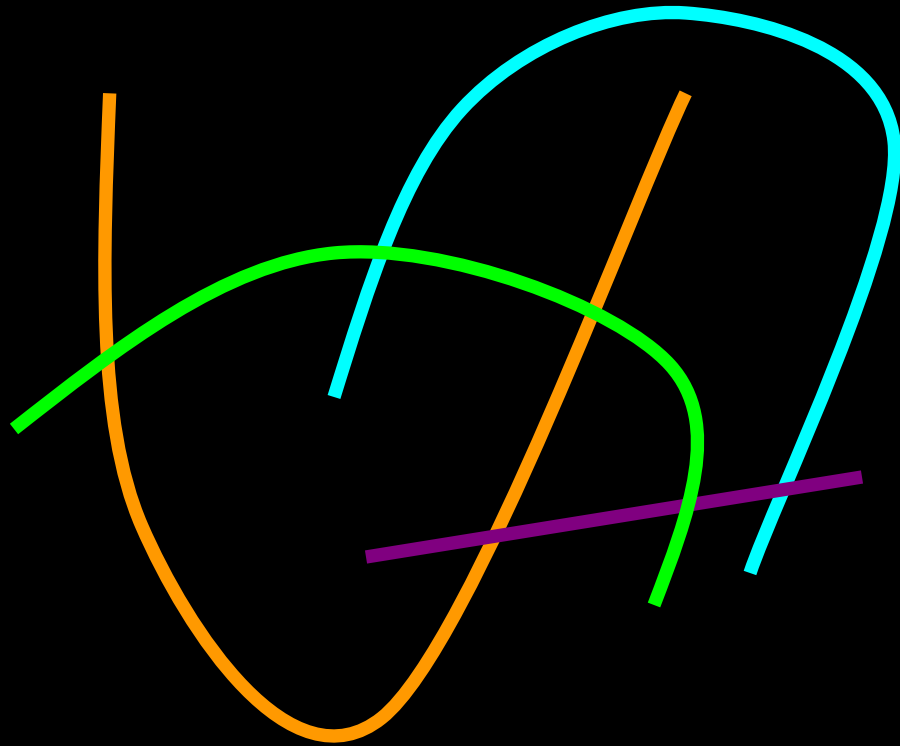
Sinden 1966

Topology of Thin Film RC Circuits



	1	2	3	4
1		X		X
2	X		X	X
3		X		X
4	X	X	X	

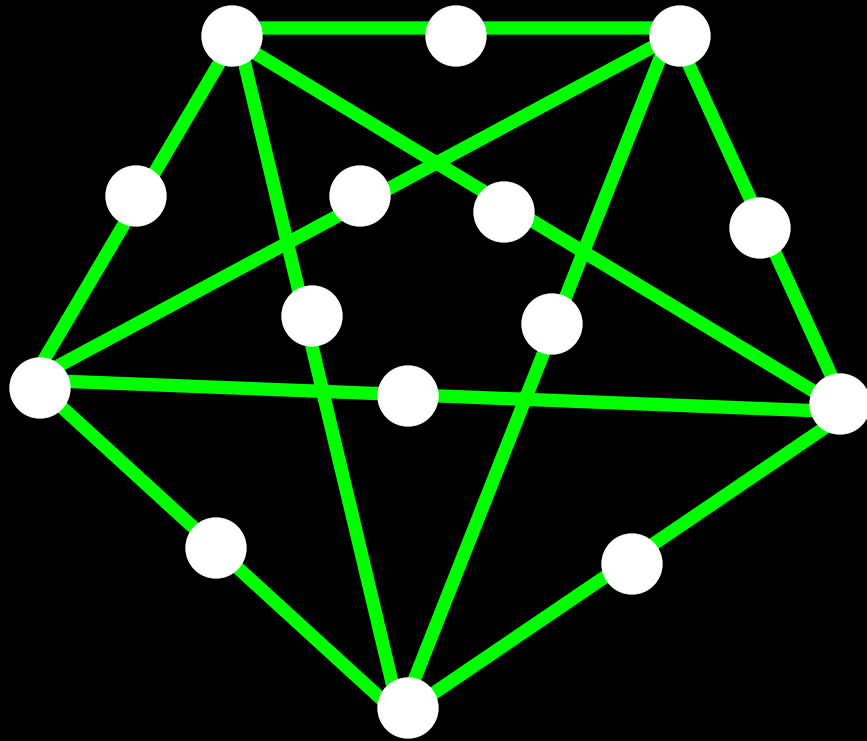
String Graph



Is G an intersection graph of a set of curves in the plane?

	1	2	3	4
1		X		X
2	X		X	X
3		X		X
4	X	X	X	

String Graphs



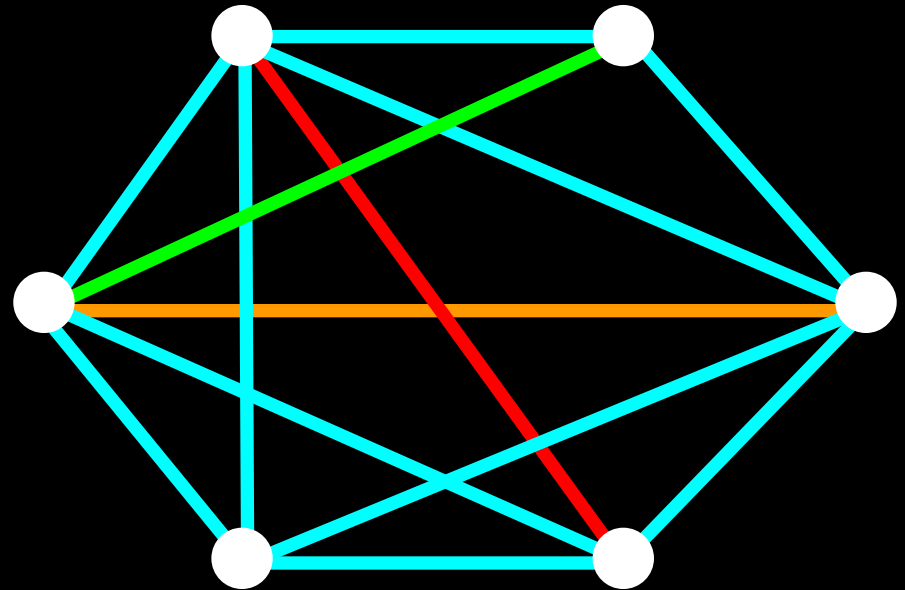
**Planar graphs are
string graphs
(Sinden, 1966)**

**Recognizing string
graphs is NP-hard
(Kratochvíl, 1991)**

**Recognizing string
graphs is decidable
(in NEXP)
(Pach, Tóth, 2000;
Schaefer, Š, 2000)**

Weak realizability

$G =$



Can G be drawn in the plane ?

- **red** edge may intersect **green** edge
- **red** edge may intersect **orange** edge
- no other pair of edges may intersect

String Weak realizability (Matoušek, Nešetřil, Thomas'88)

Weak realizability

Input:

- **Graph G**
- **set R of pairs of edges**

Output:

Is there a drawing of G in the plane such that only pairs from R may intersect?

(e.g. $R=0$ corresponds to planarity testing)

Weak realizability

NP-hard (Kratochvíl '91)

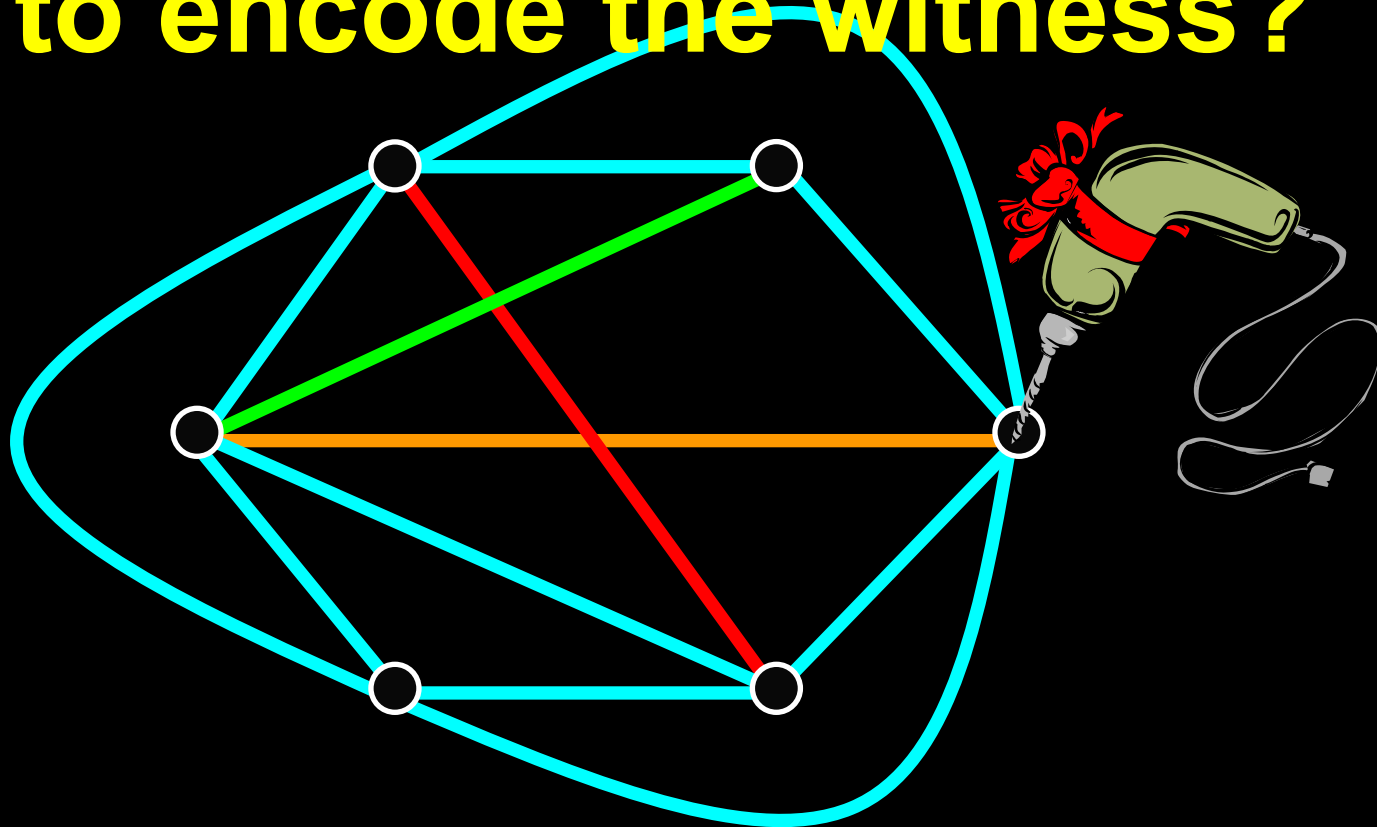
NEXP (Pach, Tóth '00; Schaefer, Š '00)



Theorem: If there is a drawing realizing (G, R) then there is a drawing with at most m^2 intersections where m is the number of edges of G .

The Theorem is tight (Kratochvíl, Matoušek '91)

How to encode the witness?



edge

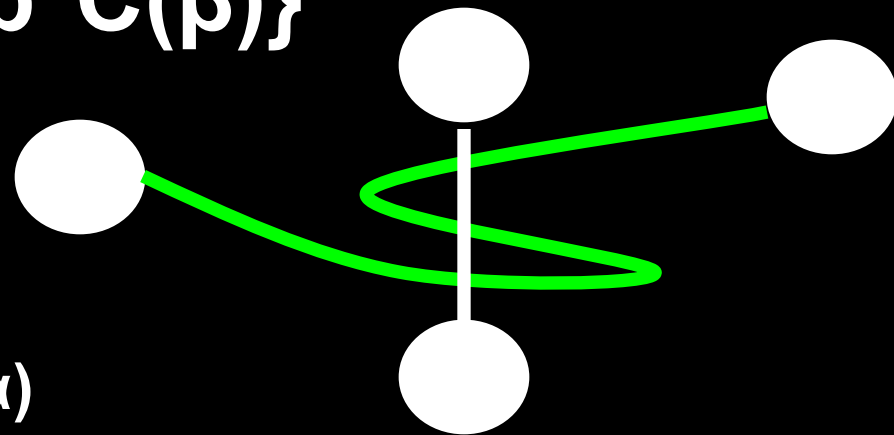
properly embedded arc (parc)

isotopy rel endpoints = continuous
deformations not moving endpoints

Intersection number $i(\alpha, \beta)$ of two parcs α, β

$$\min\{|a - b| ; a \in C(\alpha), b \in C(\beta)\}$$

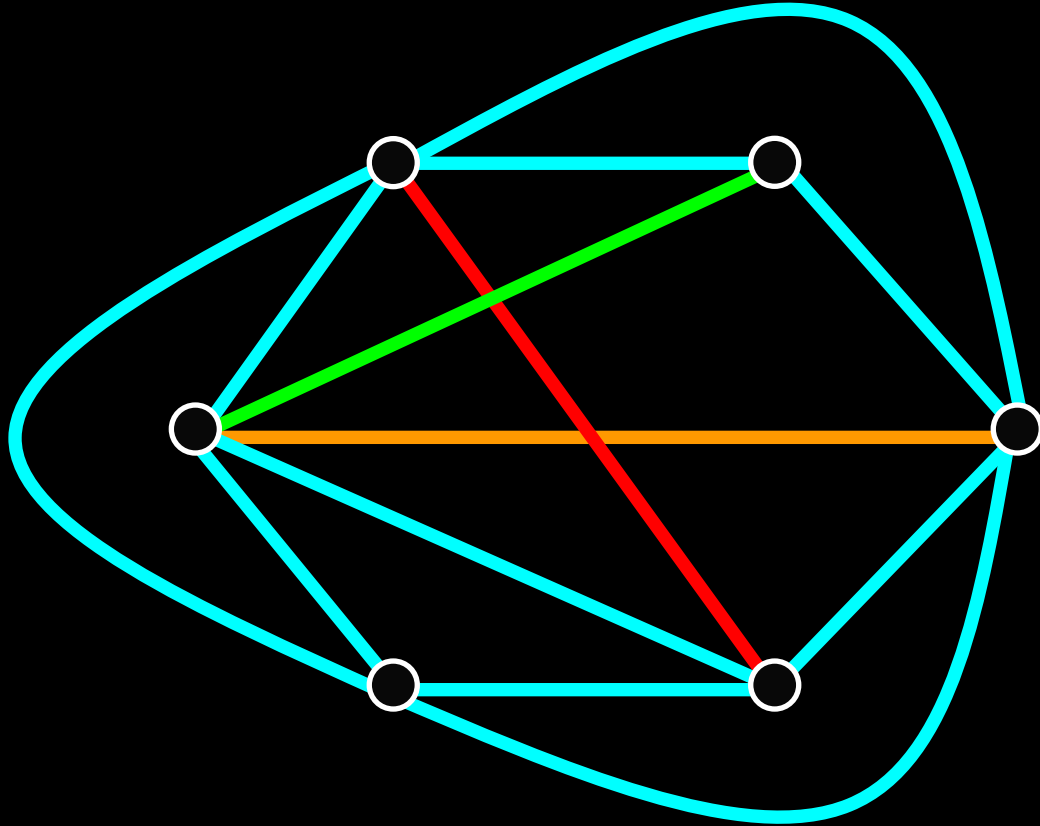
(set of curves isotopic to α)



Lemma:

On an orientable surface any collection of parcs can be redrawn so that any two parcs α, β intersect at most $i(\alpha, \beta)$ times.

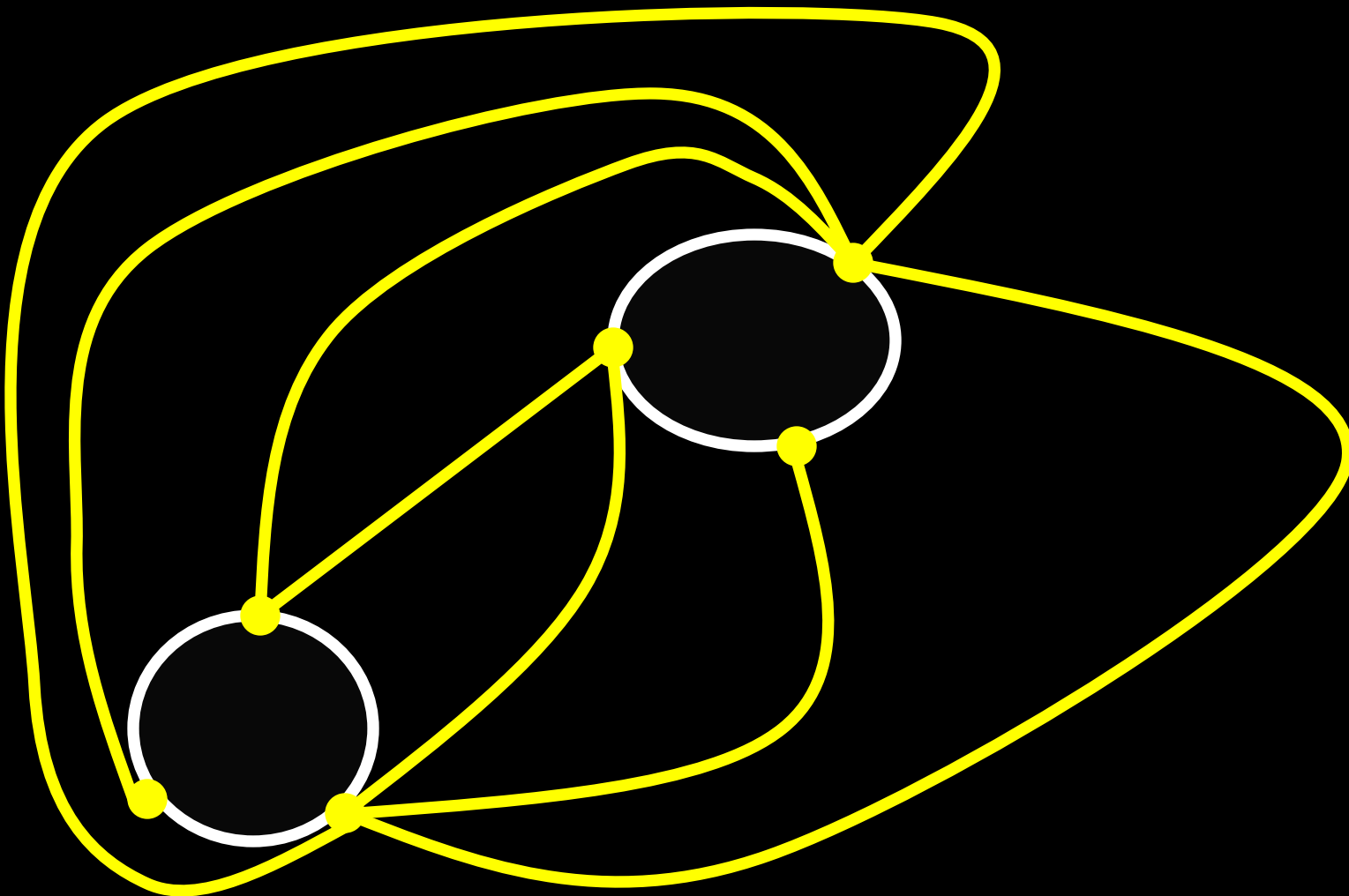
The proof of weak realizability



- encode the properly embedded arcs (up to isotopy)
- for each pair α, β not in R check $i(\alpha, \beta) = 0$

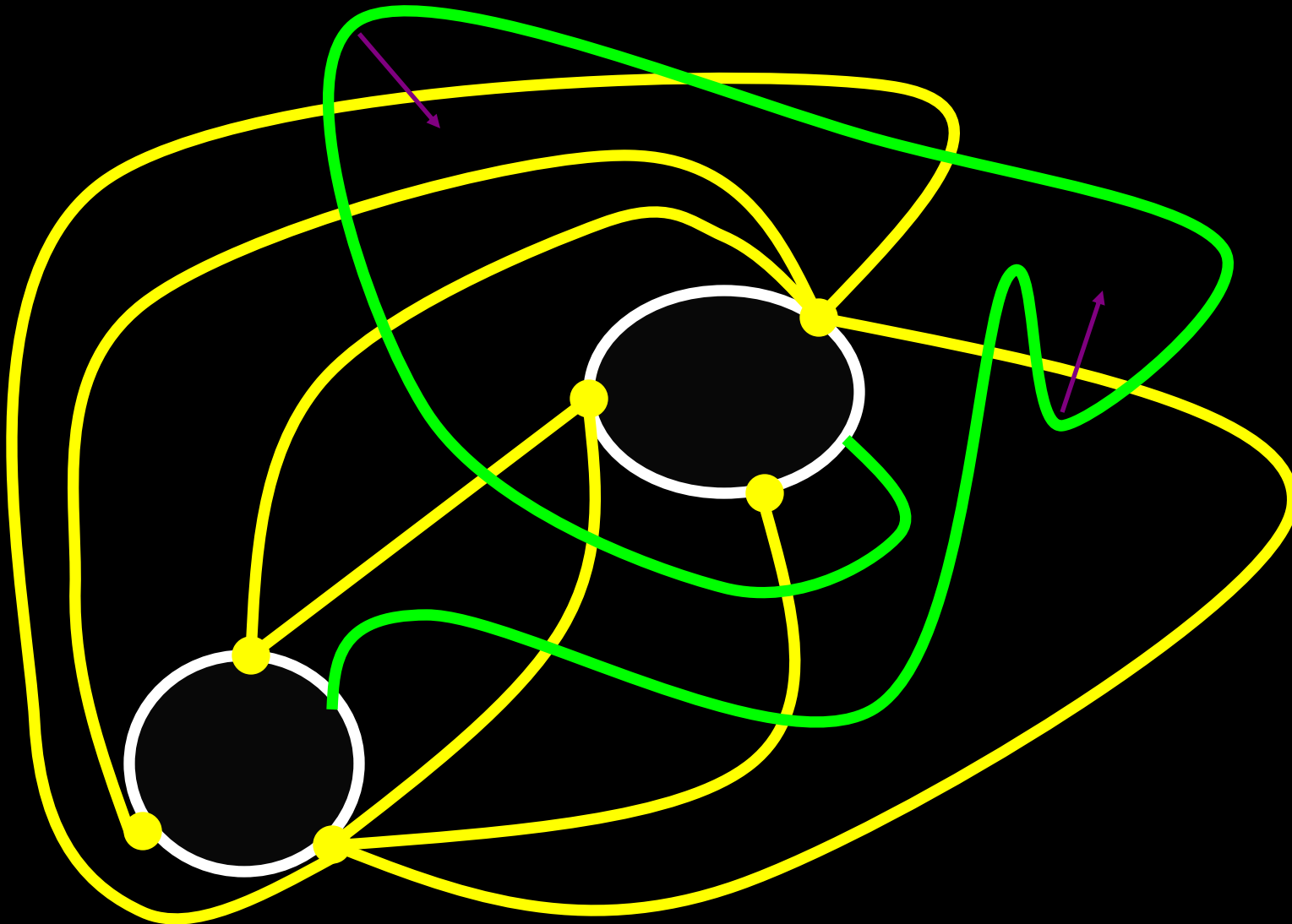
Encoding the parcs

1) A triangulation T of M



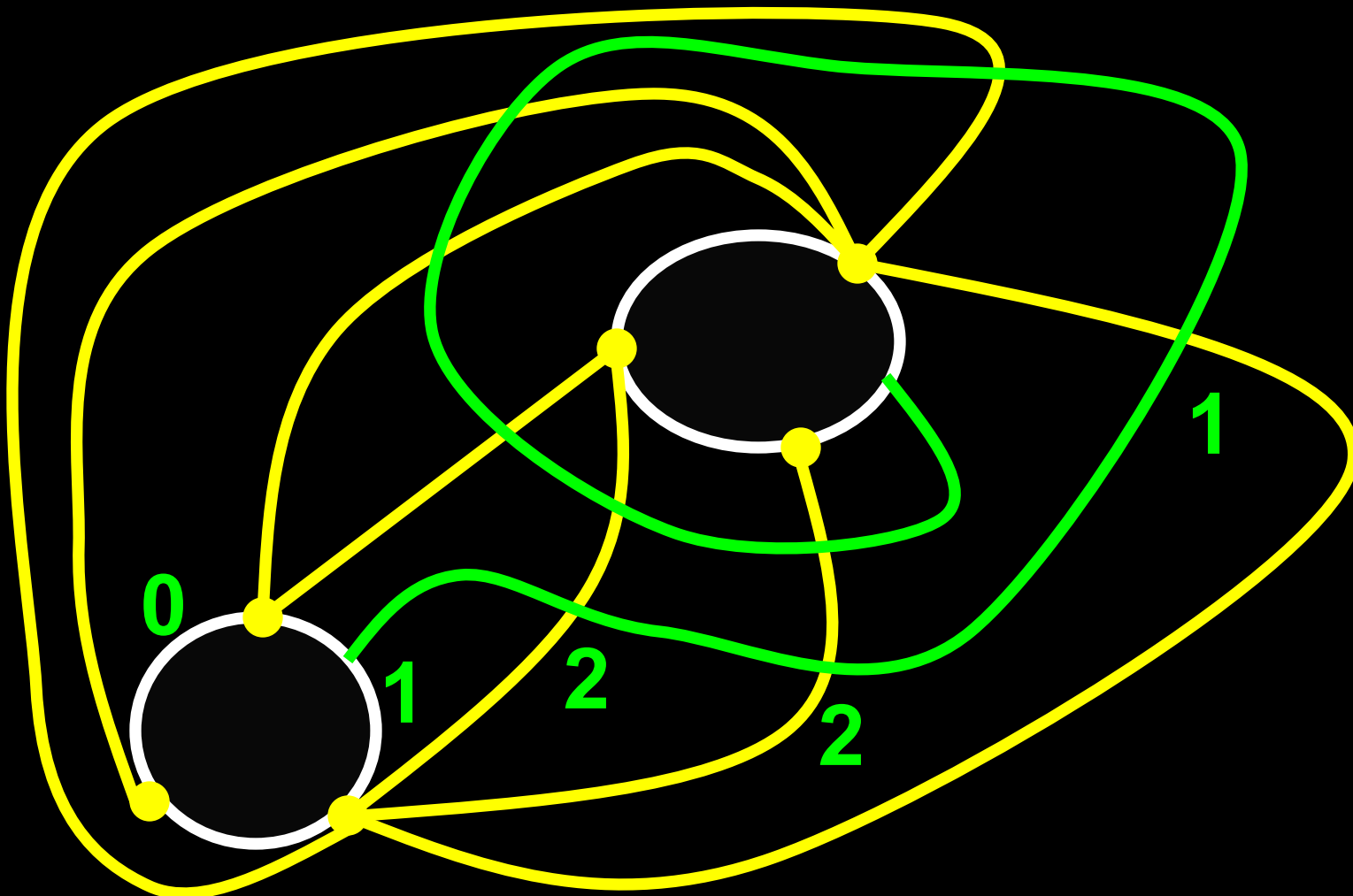
Encoding the parcs

2) Normalization of the parc w.r.t. T



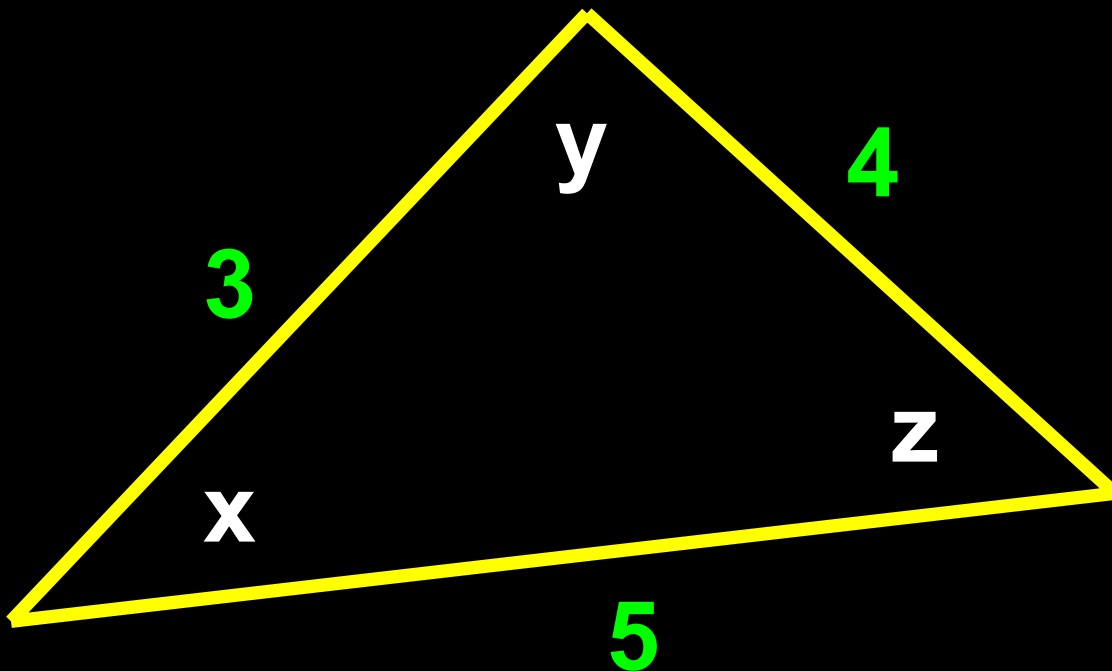
Encoding the parcs

3) Compute normal coordinates



Encoding the parcs

Parcs having the same normal coordinates are isotopic rel boundary.

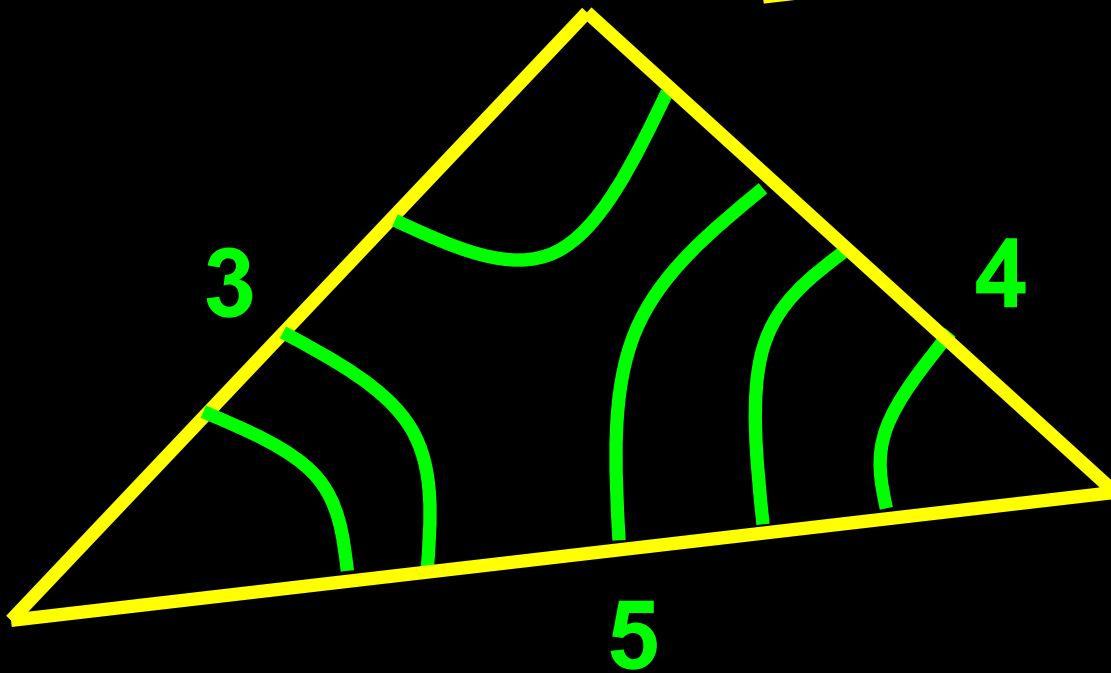
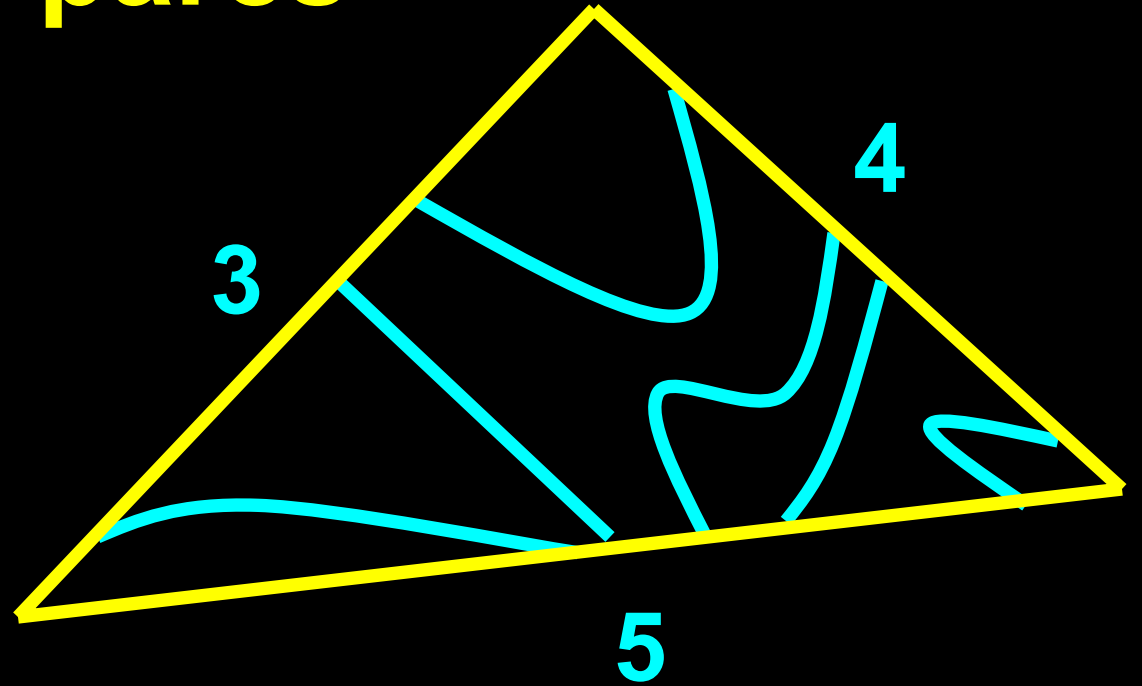


$$x+y=3$$

$$x+z=5$$

$$y+z=4$$

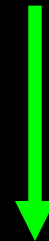
Encoding the parcs



Encoding the parcs

Is it polynomial ?

construct a weak realizability problem including the triangulation and use



Theorem: If there is a drawing realizing (G,R) then there is a drawing with at most m^2 intersections where m is the number of edges of G .

Word equations

$$xayxb = axbxy$$

a solution

$$x = aaaa$$

$$y = b$$

x, y – variables
 a, b – constants


Word equations with given lengths

$$xayxb = axbxy$$

$$|x| = 4$$

$$|y| = 1$$

The size of the bit representation of the numbers counts to the size of the input



Word equations

NP-hard

in PSPACE (Plandowski '99)

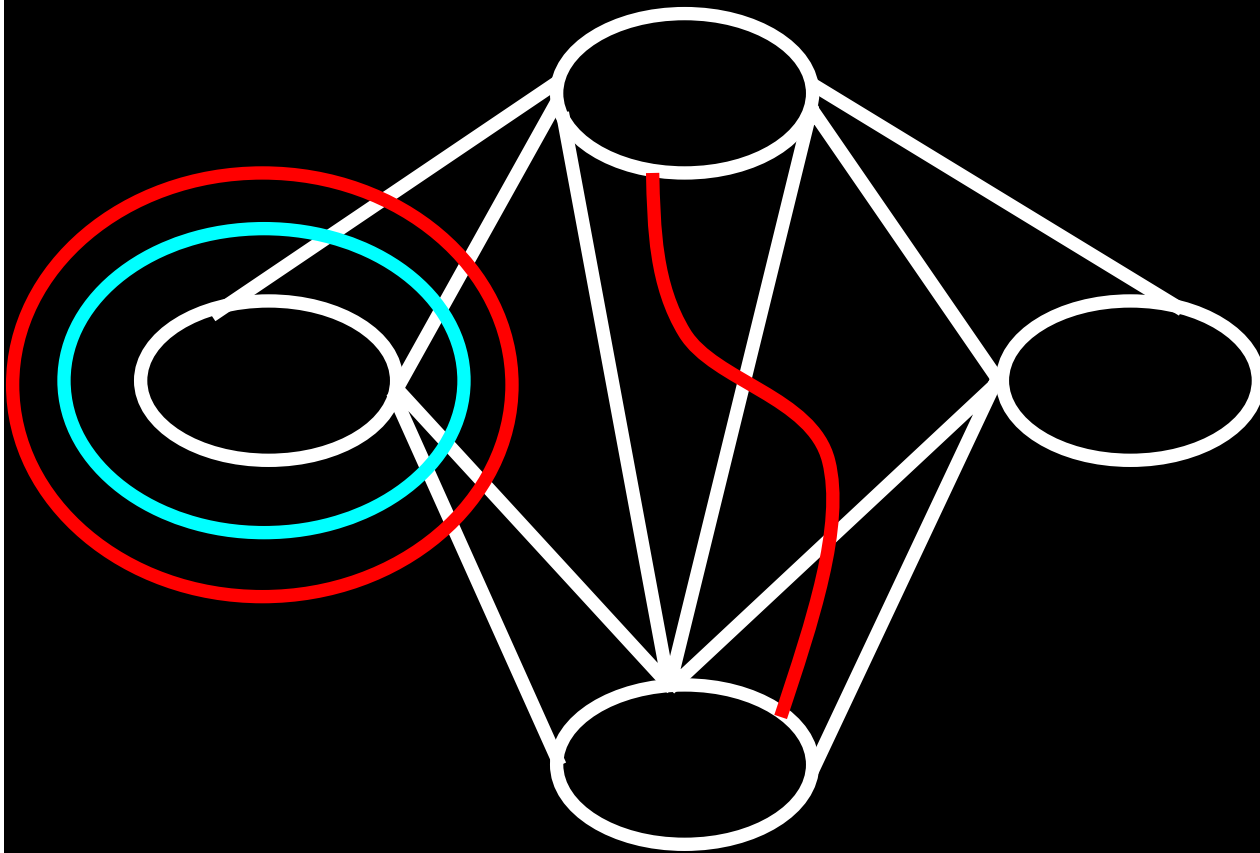
Word equations with given lengths

in P (Plandowski, Rytter '98)

**the lexicographically smallest solution
given by a straight line program**

Coloring components of a curve

normal coordinates – can encode any embedded collection of closed curves and arcs (=curve)



$$x+y=a$$

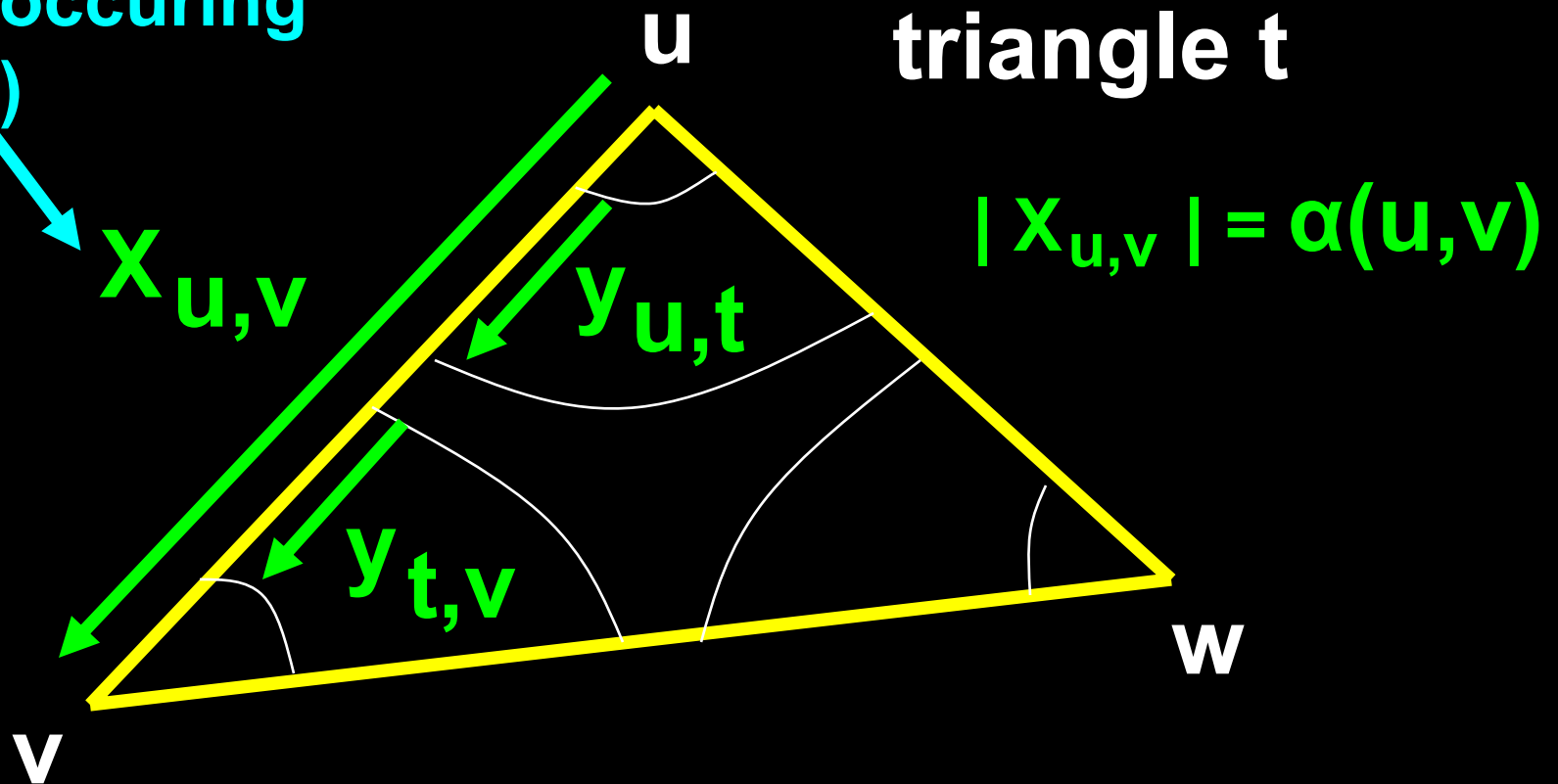
$$x+z=b$$

$$y+z=c$$

Coloring components of a curve

normal coordinates – can encode any embedded collection of closed curves and arcs (=curve)

colors occurring
on (u,v)



for edges from T M equation $X_{u,v} = a, b, \dots$

The proof of weak realizability

- encode the properly embedded arcs (up to isotopy)
- for each pair α, β not in R check $i(\alpha, \beta) = 0$

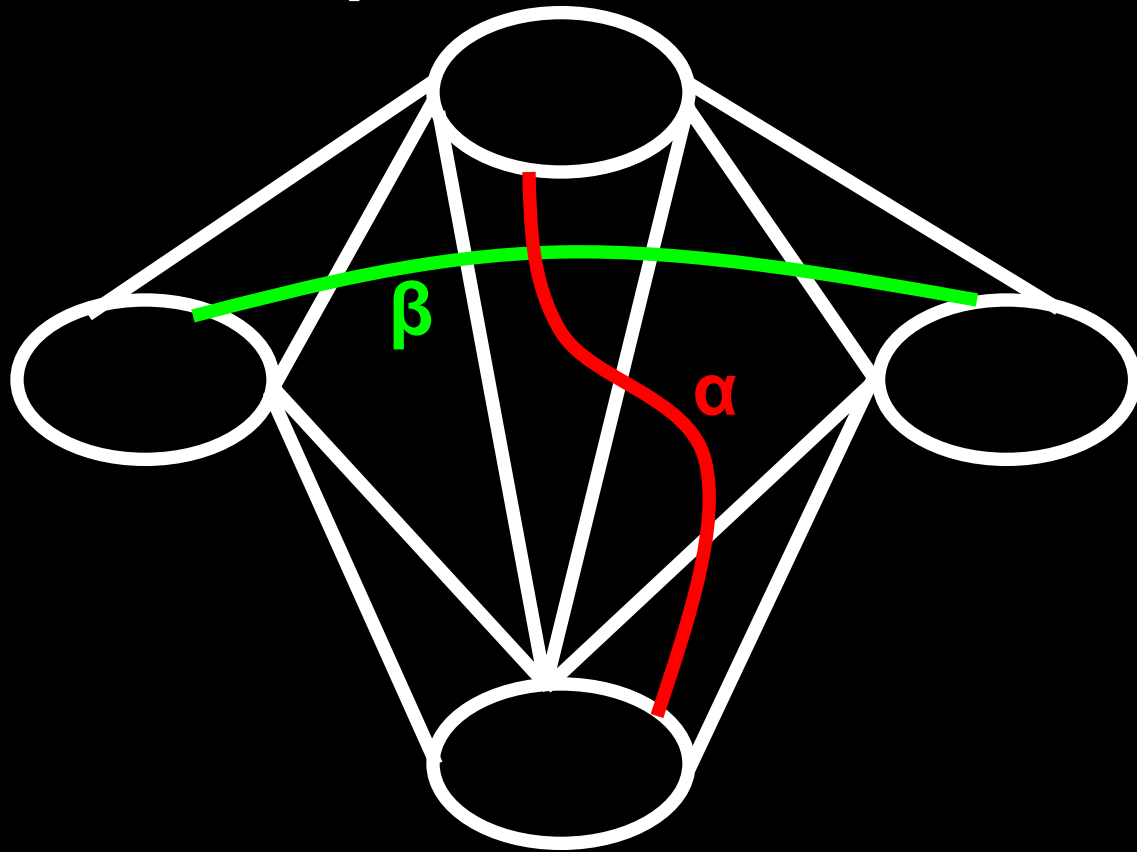
Do coordinates of α encode a parc?

Are parcs α, β isotopically disjoint?

- check that both α, β are parcs
- color one component of $\alpha + \beta$ by “b”
- They are disjoint **iff** the component is either α or β

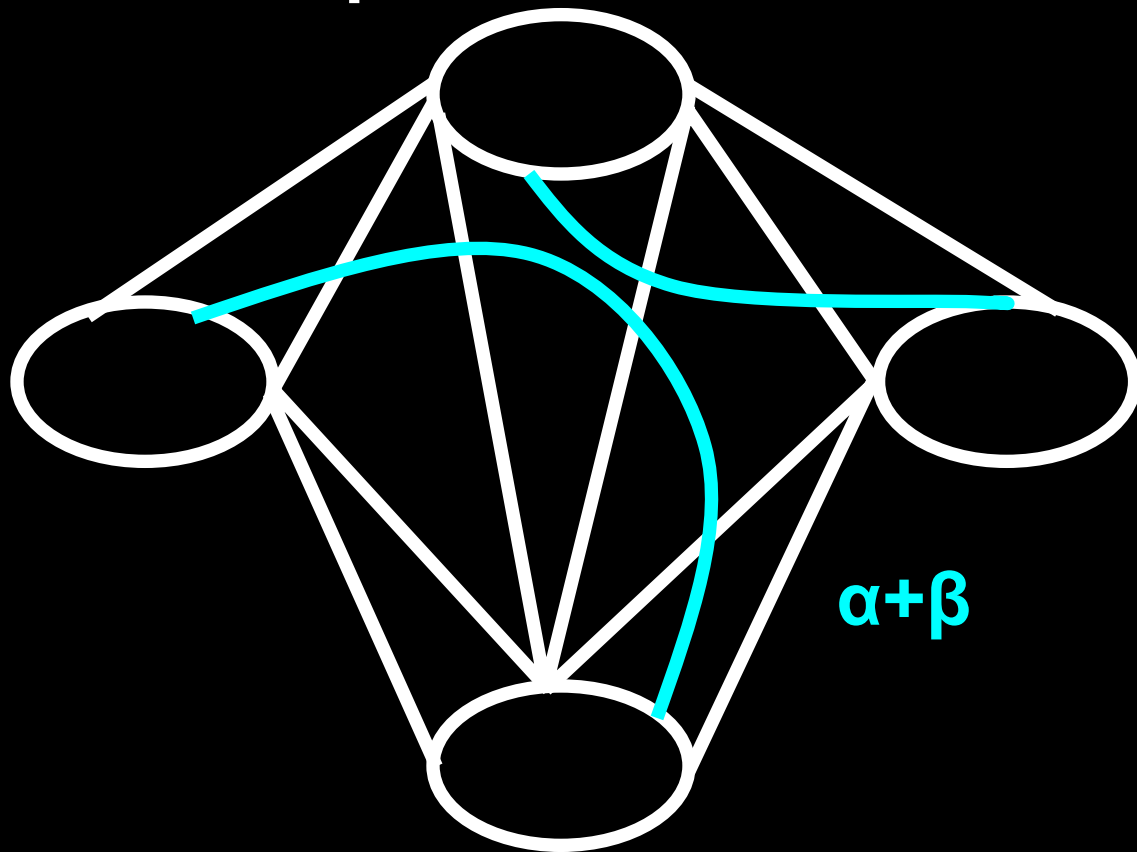
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Are parcs α , β isotopically disjoint?

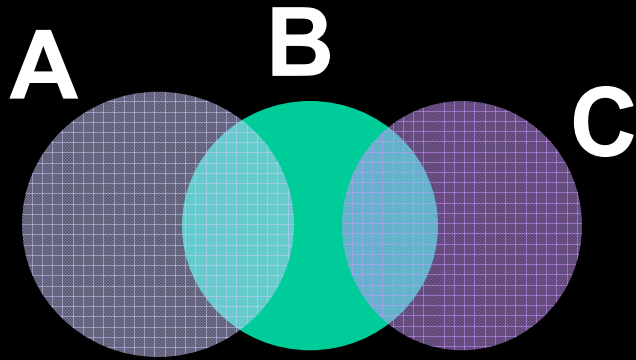
- check that both α , β are parcs
- color one component of $\alpha+\beta$ by “b”
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Consequences + other results

pairwise crossing number NP

existential theory of diagrams
(topological inference) NP



A intersects B
B intersects C
A is disjoint from C

weak realizability on different surfaces

Can be done in NP?